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Statistical properties of a solvable three-boson squeeze operator model

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Abstract. In the present paper we introduce a new squeeze operator, which is related to the time-dependent evolution operator for Hamiltonian representing mutual interaction between three different modes. Squeezing phenomenon as well as the variances of the photon-number sum and difference are considered. Moreover, Glauber second-order correlation function is discussed, besides the quasiprobability distribution function and phase distribution for different states. The joint photon-number distribution is also reported.

PACS. 42.50.-p Quantum optics – 42.50.Dv Nonclassical field states; squeezed, antibunched, and sub-Poissonian states; operational definitions of the phase of the field; phase measurements

1 Introduction

Since the beginning of the last decade considerable efforts have been paid to the non-classical phenomena that partially characterize quantum mechanically a radiation field without a classical analogue; one of these phenomena is called squeezing of vacuum fluctuations. These efforts are motivated by the potential applications of squeezed states in optical communications and ultra-sensitive detection systems, where the squeezed states of light have been generated efficiently, for example, by optical parametric down-conversion. Further in order to produce sub-Poissonian photoelectron counting statistics, one can use the correlated states that have been generated in parametric down-conversion. The squeezed states of light have also been investigated in a resonance fluorescence systems of one atom [1–3] and $N > 1$ cooperative atoms, or $N > 1$ Rydberg atoms in optical cavities [4–6], where the thermal noise in input fields is always large. Furthermore, theoretical predictions have shown that squeezing of quantum fluctuations can occur in a variety of non-linear processes, such as parametric amplifications, four wave mixing, and non-linear propagation of light. Since the correlation between states of light represents the natural product of two non-linear optical processes, it is interesting to mention the paper [7] discussing the connection between quantum optical systems using a model of parametric down-conversion, where two electromagnetic field modes become tightly correlated through their nonlinear interaction. To produce squeezed states, there are two familiar squeeze operators. The first one is the two-photon squeeze operator

$$\hat{S}(\xi) = \exp(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2}), \tag{1.1}$$

which produces a more general quadratic generator of squeezed states. The action of this operator on the one-mode vacuum state is to produce the one-mode pure state

$$\hat{S}(\xi)|0\rangle = (\mathrm{sech}\, r)^{1/2} \sum_{n=0}^{\infty} \left(\frac{1}{2} \mathrm{e}^{\mathrm{i}\theta} \tanh r\right)^n \frac{\sqrt{2n!}}{n!} |2n\rangle, \tag{1.2}$$

where $\xi = r/2 \exp(-\mathrm{i}\theta)$.

The second squeeze operator is the unitary operator effecting the Bogoliubov transformation, namely,

$$\hat{S}(\xi) = \exp(\xi^* \hat{a}^\dagger \hat{b}^\dagger - \xi \hat{a}\hat{b}), \tag{1.3}$$

which can produce the correlated states of two field modes ($\hat{a}$ and $\hat{b}$). The action of this operator on the two-mode vacuum state is to produce the two-mode pure state

$$\hat{S}(\xi)|0,0\rangle = \mathrm{sech}\,(r/2) \sum_{n=0}^{\infty} \tanh^n(r/2) \mathrm{e}^{\mathrm{i}n\theta} |n_a, n_b\rangle. \tag{1.4}$$

In the previous communications [8] we have introduced three different types of squeeze operators, all of them are just a generalization of the above squeeze operator model. The main concentration of the previous work was on the studying of the statistical properties of these operators related to the number state, coherent state, and thermal state. We have also extended our discussions to include the

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quasi-probability distribution functions, for more details see [8].

In the present paper we shall generalize the previous works and introduce a highly correlated multidimensional squeeze operator. This operator is the time-dependent evolution operator for Hamiltonian representing mutual interaction between three different modes. Our plan to study the properties of this operator is as follows: in Section 2 we develop the operator form and discuss some of its properties. In Section 3 we derive the basic relations for three-mode squeeze operator and wave function in terms of coherent representation, further we will discuss three-mode photon-number sum and photon-number difference as well as squeezing phenomenon. Section 4 is devoted to discuss sub-Poissonian phenomenon for three-mode squeezed coherent and number states. In Section 5 we include results for quasidistribution functions, and finally we summarize main conclusions in Section 6.

2 Operator formalism

In this section we shall introduce the operator model and discuss some of its properties. This operator consists of three modes in interaction and it represents the time-dependent evolution operator of the interaction part of the Hamiltonian [9,10] determined as

$$\frac{H}{\hbar} = \sum_{j=1}^{3} \omega_j \hat{a}_j^{\dagger}\hat{a}_j - \mathrm{i}\lambda_1(\hat{a}_1\hat{a}_2\mathrm{e}^{\mathrm{i}(\omega_1+\omega_2)t} - \text{h.c.}) - \mathrm{i}\lambda_2(\hat{a}_1\hat{a}_3\mathrm{e}^{\mathrm{i}(\omega_1+\omega_3)t} - \text{h.c.}) - \mathrm{i}\lambda_3(\hat{a}_2\hat{a}_3^{\dagger}\mathrm{e}^{\mathrm{i}(\omega_2-\omega_3)t} - \text{h.c.}), \tag{2.1}$$

where $\hat{a}_j$ and $\hat{a}_j^{\dagger}$ satisfy the commutation relations

$$[\hat{a}_k, \hat{a}_j^{\dagger}] = \delta_{kj}, \quad \delta_{kj} = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases} \tag{2.2}$$

and ω_j are the fields frequencies and λ_j are the effective intermodal coupling constants. In fact equation (2.1) describes a two-photon parametric coupling of modes 1 and 2, and 1 and 3 (two photons are simultaneously created or annihilated in both the quantum modes through the interaction with classical pumping mode), and linear interaction of modes 2 and 3. By introducing the transformation $\hat{A}_j = \hat{a}_j \exp(\mathrm{i}\omega_j t)$, the Hamiltonian (2.1) may be transformed into the interaction picture,

$$\frac{H}{\hbar} = -\mathrm{i}\lambda_1(\hat{A}_1\hat{A}_2 - \hat{A}_1^{\dagger}\hat{A}_2^{\dagger}) - \mathrm{i}\lambda_2(\hat{A}_1\hat{A}_3 - \hat{A}_1^{\dagger}\hat{A}_3^{\dagger}) - \mathrm{i}\lambda_3(\hat{A}_3^{\dagger}\hat{A}_2 - \hat{A}_3\hat{A}_2^{\dagger}). \tag{2.3}$$

It is important mentioning that the interaction (2.3) can be established in a bulk nonlinear crystal exhibiting the second-order nonlinear properties in which three dynamical modes of frequencies $\omega_1, \omega_2, \omega_3$ are induced by three beams from lasers of these frequencies. When pumping this crystal by means of the corresponding strong coherent pump beams, as indicated in the Hamiltonian, we can approximately fulfill the phase-matching conditions for the corresponding processes, in particular if the frequencies are close each other (biaxial crystals may be helpful in such an arrangement). Also a possible use of quasi-phase matching may help in the realization, which is, however, more difficult technologically [11]. Another possibility to realize such interaction is to use a nonlinear directional coupler which is composed of two optical waveguides fabricated from some nonlinear material described by the quadratic susceptibility $\chi^{(2)}$. Modes 1 and 2 propagate in the first waveguide and 1 and 3 in the second waveguide (auxiliary device such as bandgap quantum coupler [12] or a set of mirrors can be used to generate two identical modes (mode 1) in each waveguide). The interactions between the modes in the same waveguide are established by strong pump coherent light. In this case the coupling constants λ_1 and λ_2 are proportional to the second order susceptibility $\chi^{(2)}$ of the medium and they also include the amplitudes of the pump. The linear coupling between modes 2 and 3 is established through the evanescent waves [13].

For completeness the time-evolution operator of equation (2.3) is

$$\exp\left(-\mathrm{i}\frac{H}{\hbar}t\right) = \exp[\lambda_1 t(\hat{A}_1^{\dagger}\hat{A}_2^{\dagger} - \hat{A}_1\hat{A}_2) + \lambda_2 t(\hat{A}_1^{\dagger}\hat{A}_3^{\dagger} - \hat{A}_1\hat{A}_3) + \lambda_3 t(\hat{A}_3\hat{A}_2^{\dagger} - \hat{A}_3^{\dagger}\hat{A}_2)], \tag{2.4}$$

which can be identified well with a time-dependent three-mode squeeze operator

$$\hat{S}(\underline{r}) = \exp[r_1(\hat{A}_1^{\dagger}\hat{A}_2^{\dagger} - \hat{A}_1\hat{A}_2) + r_2(\hat{A}_1^{\dagger}\hat{A}_3^{\dagger} - \hat{A}_1\hat{A}_3) + r_3(\hat{A}_3\hat{A}_2^{\dagger} - \hat{A}_3^{\dagger}\hat{A}_2)], \tag{2.5}$$

where $r_j = \lambda_j t$, with $0 \leq r_j < \infty$, $j = 1, 2, 3$ and $\underline{r} = (r_1, r_2, r_3)$. It is evident that this squeeze operator must involve two different squeezing mechanisms (terms involving r_1 and r_2) and then it would be more complicated than squeezing operators that have appeared in the literature earlier [14–18]. Now if we set

$$\hat{A} = (\hat{A}_1^{\dagger}\hat{A}_2^{\dagger} - \hat{A}_1\hat{A}_2), \tag{2.6a}$$

$$\hat{B} = (\hat{A}_1^{\dagger}\hat{A}_3^{\dagger} - \hat{A}_1\hat{A}_3), \tag{2.6b}$$

$$\hat{C} = (\hat{A}_2\hat{A}_3^{\dagger} - \hat{A}_3\hat{A}_2^{\dagger}), \tag{2.6c}$$

then we have the following commutation relations

$$[\hat{A}, \hat{B}] = -\hat{C}, \tag{2.7a}$$

$$[\hat{B}, \hat{C}] = \hat{A}, \tag{2.7b}$$

$$[\hat{C}, \hat{A}] = \hat{B}. \tag{2.7c}$$

We may conclude that the squeeze operator (2.5) involves correlations and can be regarded as the exponential of linear combination of three generators, which are closed under the commutation relations (2.7), and it represents the $su(1,1)$ generalized coherent state. This in fact encouraged us to study such a type of operators where an exact solution can be found. It is worth mentioning that the three-boson interaction can also provide $su(2)$ generalized coherent state (Bloch state) by considering the following operator

$$\hat{S}(\underline{R}) = \exp[R_1(\hat{A}_1^\dagger \hat{A}_2 - \hat{A}_1 \hat{A}_2^\dagger) + R_2(\hat{A}_1^\dagger \hat{A}_3 - \hat{A}_1 \hat{A}_3^\dagger) + R_3(\hat{A}_3^\dagger \hat{A}_2 - \hat{A}_3 \hat{A}_2^\dagger)], \quad (2.8)$$

where $\underline{R} = (R_1, R_2, R_3)$ and R_j have the same meaning as r_j. Assuming that

$$\hat{J}_x = \mathrm{i}(\hat{A}_1^\dagger \hat{A}_2 - \hat{A}_1 \hat{A}_2^\dagger), \quad (2.9a)$$

$$\hat{J}_y = \mathrm{i}(\hat{A}_1^\dagger \hat{A}_3 - \hat{A}_1 \hat{A}_3^\dagger), \quad (2.9b)$$

$$\hat{J}_z = \mathrm{i}(\hat{A}_3^\dagger \hat{A}_2 - \hat{A}_2^\dagger \hat{A}_3), \quad (2.9c)$$

one can easily verify that these operators (2.9) are the generators of the $su(2)$ Lie algebra since they satisfy the commutation rules

$$[\hat{J}_x, \hat{J}_y] = \mathrm{i}\hat{J}_z, \quad (2.10a)$$

$$[\hat{J}_y, \hat{J}_z] = \mathrm{i}\hat{J}_x, \quad (2.10b)$$

$$[\hat{J}_z, \hat{J}_x] = \mathrm{i}\hat{J}_y. \quad (2.10c)$$

As known the unitary representation of $su(2)$ Lie algebra is labeled by the angular momentum quantum number. Nevertheless, if one considers an operator related to the three-boson interaction, but including three squeezing mechanisms, *i.e.* in the operator (2.5) replacing the term involving r_3 by $(\hat{A}_3^\dagger \hat{A}_2^\dagger - \hat{A}_3 \hat{A}_2)$, then this operator is also of interest, however, the mathematics related to this type of operator is rather complicated. Moreover, it cannot produce neither $su(1,1)$ nor $su(2)$ generalized coherent states.

As is well-known quantum correlation between different quantum mechanical systems can give rise to nonclassical effects in operators that act in the space of both the systems even if the individual operator of the single system does not exhibit such effects, *i.e.* if the measurement of an observable of the first system (say), for correlated system, is performed, this projects the other system into new states; otherwise the systems are uncorrelated. Thus if we consider the total density operator of our quantum system to be $\hat{\rho}$ to describe three of correlated quantum systems $\hat{A}_j, j = 1, 2, 3$, then the reduced density matrix of any system can be obtained by

$$\hat{\rho}_{\hat{A}_j} = \mathrm{Tr}_{\hat{A}_k, \hat{A}_s} \hat{\rho}, \quad (2.11a)$$

where j, k, s take different values 1, 2, 3 and $\mathrm{Tr}_{\hat{A}_k, \hat{A}_s}$ denotes two trace operations performed simultaneously. Hence the $\hat{A}_j$ systems are correlated if the measurement of an observable of $\hat{A}_1$ system, say, projects $\hat{A}_2$ and $\hat{A}_3$ systems into new states. Nevertheless, if the three system density operators can be written in the factorized form,

$$\hat{\rho} = \hat{\rho}_{\hat{A}_1} \otimes \hat{\rho}_{\hat{A}_2} \otimes \hat{\rho}_{\hat{A}_3}, \quad (2.11b)$$

then the systems are uncorrelated.

Squeezing property is the important phenomenon distinguishing well mechanism of correlation of systems, where squeezing can occur in combination of the quantum mechanical systems described by operators $\hat{A}_1, \hat{A}_2$ and $\hat{A}_3$, even if single systems are not themselves squeezed. In fact, the ideas that quantum correlations can give rise to squeezing in the combination of system operators has been shown true for multimode squeezed states of light [15,19–22] and for dipole fluctuations in multimode squeezed states [23].

3 Properties of the correlated quantum systems

Squeezed state of light is distinguishable by long-axis variance of noise ellipse for one of its quadratures in phase-space. This property is connected with the pairwise nature of the unitary operator (1.1) under which the initial state evolves. This operator exhibits some well-known basic relations summarized in the literature [14]. We will introduce similar relations corresponding to the three-mode squeeze operator (2.5), and then we shall use them to deduce the three-mode photon-number sum and difference as well as the wave function in the coherent state representation.

The squeeze operator (2.5) provides a Bogoliubov transformation of the annihilation and creation operators that mixe the three modes as

$$\bar{A}_1 \equiv \hat{S}^{-1}(\underline{r}) \hat{A}_1 \hat{S}(\underline{r}) = \hat{A}_1 f_1 + \hat{A}_2^\dagger f_2 + \hat{A}_3^\dagger f_3, \quad (3.1a)$$

$$\bar{A}_2 \equiv \hat{S}^{-1}(\underline{r}) \hat{A}_2 \hat{S}(\underline{r}) = \hat{A}_2 g_1 + \hat{A}_3 g_2 + \hat{A}_1^\dagger g_3, \quad (3.1b)$$

$$\bar{A}_3 \equiv \hat{S}^{-1}(\underline{r}) \hat{A}_3 \hat{S}(\underline{r}) = \hat{A}_3 h_1 + \hat{A}_1^\dagger h_2 + \hat{A}_2 h_3, \quad (3.1c)$$

where

$$f_1 = \cosh\mu + \frac{2r_3^2}{\mu^2} \sinh^2\left(\frac{\mu}{2}\right), \quad (3.2a)$$

$$f_2 = \frac{r_1}{\mu} \sinh\mu - \frac{2r_2 r_3}{\mu^2} \sinh^2\left(\frac{\mu}{2}\right), \quad (3.2b)$$

$$f_3 = \frac{r_2}{\mu} \sinh\mu + \frac{2r_1 r_3}{\mu^2} \sinh^2\left(\frac{\mu}{2}\right), \quad (3.2c)$$

and

$$g_1 = \cosh\mu - \frac{2r_2^2}{\mu^2} \sinh^2\left(\frac{\mu}{2}\right), \quad (3.3a)$$

$$g_2 = \frac{r_3}{\mu} \sinh\mu + \frac{2r_1 r_2}{\mu^2} \sinh^2\left(\frac{\mu}{2}\right), \quad (3.3b)$$

$$g_3 = \frac{r_1}{\mu} \sinh\mu + \frac{2r_2 r_3}{\mu^2} \sinh^2\left(\frac{\mu}{2}\right), \quad (3.3c)$$

whereas

$$h_1 = \cosh\mu - \frac{2r_1^2}{\mu^2} \sinh^2\left(\frac{\mu}{2}\right), \quad (3.4a)$$

$$h_2 = \frac{r_2}{\mu} \sinh\mu - \frac{2r_1 r_3}{\mu^2} \sinh^2\left(\frac{\mu}{2}\right), \quad (3.4b)$$

$$h_3 = \frac{-r_3}{\mu} \sinh\mu + \frac{2r_1 r_2}{\mu^2} \sinh^2\left(\frac{\mu}{2}\right), \quad (3.4c)$$

where $\mu = \sqrt{r_1^2 + r_2^2 - r_3^2}$ and $r_3^2 < r_1^2 + r_2^2$.

The commutation relations (2.2) under the transformations (3.1) hold also for the operators $\bar{A}_j$. Using these transformations we can easily calculate the statistical properties for each mode. It is worth to mention that the corresponding expressions for the single mode squeezed operator [14] and for two-mode squeezed operator [16] can be obtained from (3.1) by taking $r_2 = r_3 = 0$, and $\hat{A}_3 \to \hat{0}$ together with $\hat{A}_2 \to \hat{A}_1$ for only single mode case. Furthermore, the inverse transformation of equations (3.1) can be written as

$$\hat{A}_1 = f_1\bar{A}_1 - g_3\bar{A}_2^\dagger - h_2\bar{A}_3^\dagger, \tag{3.5a}$$

$$\hat{A}_2^\dagger = -f_2\bar{A}_1 + g_1\bar{A}_2^\dagger + h_3\bar{A}_3^\dagger, \tag{3.5b}$$

$$\hat{A}_3^\dagger = -f_3\bar{A}_1 + g_2\bar{A}_2^\dagger + h_1\bar{A}_3^\dagger. \tag{3.5c}$$

We may point out that a strong correlation is built up between the three modes described by squeeze operator (2.5). This is quite obvious for the case of the parametric amplification when two mode waves are mixed to generate a third wave *via* a nonlinear medium, *e.g.* in an optical crystal with nonlinear second order susceptibility [14]. This can be demonstrated with the help of three-mode pure squeezed vacuum states $\hat{S}(\underline{r})\prod_{j=1}^3|0_j\rangle$, where $\hat{S}(\underline{r})$ is the squeeze operator (2.5). In this case the eigenstates of the three-mode photon-number difference $\hat{A}_1^\dagger\hat{A}_1 - \hat{A}_2^\dagger\hat{A}_2 - \hat{A}_3^\dagger\hat{A}_3$ correspond to zero eigenvalue, thus

$$\varDelta(\hat{A}_1^\dagger\hat{A}_1 - \hat{A}_2^\dagger\hat{A}_2 - \hat{A}_3^\dagger\hat{A}_3)^2 = 0. \tag{3.6a}$$

However, the situation will be different for three-mode photon-number sum, after minor calculations we obtain

$$\varDelta(\hat{A}_1^\dagger\hat{A}_1 + \hat{A}_2^\dagger\hat{A}_2 + \hat{A}_3^\dagger\hat{A}_3)^2 = f_1^2(f_1^2 - 1) + g_3^2(1 + g_3^2) + h_2^2(1 + h_2^2) + 2(f_1^2g_3^2 + f_1^2h_2^2 + h_2^2g_3^2). \tag{3.6b}$$

In order to see the correlation between modes we have to calculate the fluctuations for both sum and difference operators thus having

$$2\Big(\langle\hat{A}_1^\dagger\hat{A}_1\hat{A}_2^\dagger\hat{A}_2\rangle + \langle\hat{A}_1^\dagger\hat{A}_1\hat{A}_3^\dagger\hat{A}_3\rangle + \langle\hat{A}_2^\dagger\hat{A}_2\hat{A}_3^\dagger\hat{A}_3\rangle - \langle\hat{A}_1^\dagger\hat{A}_1\rangle\langle\hat{A}_2^\dagger\hat{A}_2\rangle - \langle\hat{A}_1^\dagger\hat{A}_1\rangle\langle\hat{A}_3^\dagger\hat{A}_3\rangle - \langle\hat{A}_2^\dagger\hat{A}_2\rangle\langle\hat{A}_3^\dagger\hat{A}_3\rangle\Big) = 2(f_1^2g_3^2 + f_1^2h_2^2 + h_2^2g_3^2). \tag{3.6c}$$

It is clear that this quantity has non-zero value and this is the signature of photon-number correlation between modes.

Now we show how quantum correlations between the systems can give rise to squeezing if operators act in the spaces not only corresponding to three systems (three-mode squeezing) but also to two systems (two-mode squeezing), rather than to the individual systems (single-mode squeezing). This will be done using three-mode pure squeezed vacuum states $\hat{S}(\underline{r})\prod_{j=1}^3|0_j\rangle$ as before. For this purpose, suppose we have two quadratures $\hat{X}$ and $\hat{Y}$ which are related to the conjugate electric and magnetic field operators $\hat{E}$ and $\hat{H}$, and they are defined in the standard way. Assuming that these two quadrature operators satisfy the following commutation relation

$$\left[\hat{X}, \hat{Y}\right] = C, \tag{3.7a}$$

where C is c-number specified later, the following uncertainty relation holds

$$\langle(\varDelta\hat{X})^2\rangle\langle(\varDelta\hat{Y})^2\rangle \geq \frac{|C|^2}{4}, \tag{3.7b}$$

where $\langle(\varDelta\hat{X})^2\rangle = \langle\hat{X}^2\rangle - \langle\hat{X}\rangle^2$ is the variance. Therefore, we can say that the model possesses X- or Y-quadrature squeezing if

$$S_1 = \frac{\langle(\varDelta\hat{X})^2\rangle - 0.5|C|}{0.5|C|} < 0, \quad \text{or} \quad S_2 = \frac{\langle(\varDelta\hat{Y})^2\rangle - 0.5|C|}{0.5|C|} < 0. \tag{3.8}$$

Maximum squeezing (100%) is obtained for $S_j = -1$.

For three-mode squeezing the two quadratures have the forms

$$\hat{X} = \frac{1}{2}[\hat{A}_1 + \hat{A}_2 + \hat{A}_3 + \hat{A}_1^\dagger + \hat{A}_2^\dagger + \hat{A}_3^\dagger], \tag{3.9a}$$

$$\hat{Y} = \frac{1}{2\mathrm{i}}[\hat{A}_1 + \hat{A}_2 + \hat{A}_3 - \hat{A}_1^\dagger - \hat{A}_2^\dagger - \hat{A}_3^\dagger]. \tag{3.9b}$$

Therefore, the squeezing variances in terms of three-mode pure squeezed vacuum states are

$$\langle(\varDelta\hat{X})^2\rangle = \frac{1}{4}[1 + 2(f_1^2 + g_3^2 + h_2^2) + 4(f_1h_2 + h_2g_3 + g_3f_1)], \tag{3.10a}$$

$$\langle(\varDelta\hat{Y})^2\rangle = \frac{1}{4}[1 + 2(f_1^2 + g_3^2 + h_2^2) - 4(f_1h_2 - h_2g_3 + g_3f_1)]. \tag{3.10b}$$

The expressions for the single-mode and two-mode squeezing can be obtained easily from (3.10) for two quadratures defined in a manner analogous to (3.9) by dropping the coefficients of the absent mode, *e.g.* for the first mode, single-mode squeezing can be obtained by setting $g_j = h_j = 0$ in (3.10). It should be taken into account that $C = 1/2, 1, 3/2$ corresponding to the single-mode, two-mode and three-mode squeezing, respectively. From (3.10) one can easily prove that the model cannot exhibit single-mode squeezing, *e.g.* for the first mode we have

$$S_1 = S_2 = 2(f_2^2 + f_3^2), \tag{3.11}$$

where the relations between the coefficients f_j resulting from the commutation rules of $\hat{A}_j$ have been used to get such relation. For compound modes the system can provide two-mode (only between modes $(1,2)$ and $(1,3)$)

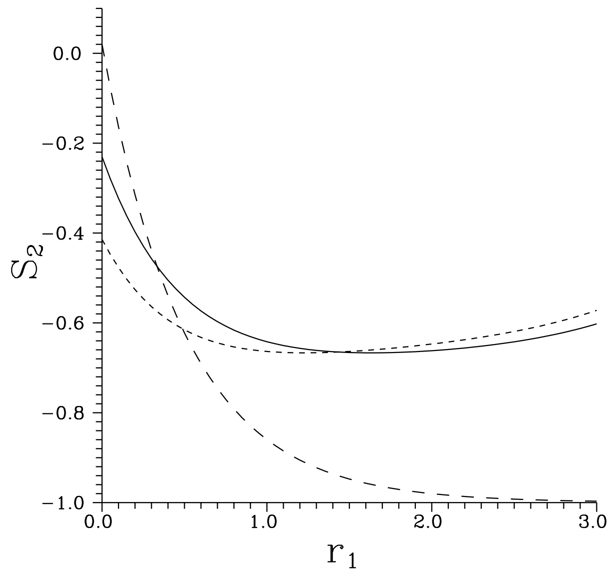


Fig. 1. S_2 against squeeze parameter r_1 for three-mode squeezing and for $r_3 = 0.1$, $r_2 = 0.1$ (solid curve) and 0.2 (short-dashed curves); and for two-mode squeezing (between modes 1 and 2, long-dashed curve) $(r_2, r_3) = (0.2, 0.1)$.

as well as three-mode squeezing in the Y-quadrature as shown in Figure 1 (for the shown values of squeeze parameters). From this figure one can observe that the behaviour of three-mode squeezing factors is smoothed in such a way that they are initially squeezed and their squeezing values reach their maximum, then they start again to decrease into unsqueezed values for large domain of r_1 (which is not shown in the figure). Also, the initial values of squeezing are sensitive to the values of the squeeze parameters (compare solid and short-dashed curves). Concerning two-mode squeezing factor (long-dashed curve) one can see it is monotonically decreasing function with lower limit -1. It is important mentioning that the squeeze operator under discussion cannot practically provide maximum squeezing (for three-mode squeezing), *i.e.* $S_2 = -1$, and this, of course, is in contrast with the single-mode and two-mode squeeze operators (1.1, 1.3) where they can display maximum squeezing for large values of squeezing parameter, however, they cannot provide squeezing initially. In conclusion it is quite obvious that squeezing can occur in the combined systems even if the individual systems are not themselves squeezed. The mechanism for this process is the correlation between the systems.

To find the wave function in the coherent state, let us introduce the following states

$$|\chi\rangle = \prod_{s=1}^{3} |\alpha_s\rangle = \exp\left(-\frac{1}{2}\sum_{s=1}^{3}|\alpha_s|^2\right) \times \sum_{n_1,n_2,n_3=0}^{\infty} \frac{\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}}{\sqrt{n_1!n_2!n_3!}}|n_1,n_2,n_3\rangle, \quad (3.12a)$$

and

$$|\bar{\chi}\rangle = \prod_{s=1}^{3} |\bar{\alpha}_s\rangle = \exp\left(-\frac{1}{2}\sum_{s=1}^{3}|\bar{\alpha}_s|^2\right) \times \sum_{m_1,m_2,m_3=0}^{\infty} \frac{\bar{\alpha}_1^{m_1}\bar{\alpha}_2^{m_2}\bar{\alpha}_3^{m_3}}{\sqrt{m_1!m_2!m_3!}}|m_1,m_2,m_3\rangle, \quad (3.12b)$$

with the following properties

$$\hat{A}_j|\chi\rangle = \alpha_j|\chi\rangle, \qquad \bar{A}_j|\bar{\chi}\rangle = \bar{\alpha}_j|\bar{\chi}\rangle. \quad (3.13)$$

From equations (3.1, 3.5, 3.12), together with equations (3.13) we have

$$\langle\chi|\bar{\chi}\rangle = f_1 \exp\Big\{\frac{1}{f_1}[(\bar{\alpha}_1\alpha_1^* + h_1\bar{\alpha}_2\alpha_2^* + g_1\bar{\alpha}_3\alpha_3^*) + (g_3\bar{\alpha}_1\bar{\alpha}_2 - f_2\alpha_1^*\alpha_2^* - h_2\bar{\alpha}_2\alpha_3^*) + (h_2\bar{\alpha}_1\bar{\alpha}_3 - f_3\alpha_1^*\alpha_3^* - g_2\bar{\alpha}_3\alpha_2^*)]\Big\} \times \exp[-\frac{1}{2}\sum_{i=1}^{3}(|\alpha_i|^2 + |\bar{\alpha}_i|^2)]. \quad (3.14)$$

Equation (3.14) represents the wave function in coherent states, where the correlations between modes are apparent. In absence of the squeezed parameters r_j, we find $\langle\chi|\bar{\chi}\rangle = 1$, which emphasizes the fact that the wave functions for $r_j > 0$ can be regarded as transition amplitudes between two different states $|\chi\rangle$ and $|\bar{\chi}\rangle$.

4 Three-mode squeezed coherent and number states

Coherent light represents the most familiar field used after discovering the laser, useful particularly in quantum optics; it satisfies the uncertainty principle with equal sign and its normalized second-order normal correlation function is always unity. Therefore we shall devote the present section to examine the Glauber second-order correlation function in terms of squeezed coherent states and squeezed number states by employing the three-mode squeeze operator (2.5). For this purpose we shall define the three-mode squeezed coherent states as

$$|\underline{\alpha}, \underline{r}\rangle \equiv \hat{S}(\underline{r})\hat{D}(\underline{\alpha})|0\rangle_1|0\rangle_2|0\rangle_3, \quad (4.1)$$

where $\hat{S}(\underline{r})$ is the three-mode squeeze operator (2.5) and $\hat{D}(\underline{\alpha})$ is the three-mode Glauber displacement operator given by

$$\hat{D}(\underline{\alpha}) = \exp\sum_{j=1}^{3}(\alpha_j\hat{A}_j^\dagger - \alpha_j^*\hat{A}_j), \quad (4.2)$$

and $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$.

The unitarity of the operator (2.5) provides that three-mode squeezed coherent states are not orthonormal but they are complete in the following sense

$$\hat{1} = \frac{1}{\pi^3}\iiint |\underline{\alpha},\underline{r}\rangle\langle\underline{r},\underline{\alpha}| \mathrm{d}^2\alpha_1 \mathrm{d}^2\alpha_2 \mathrm{d}^2\alpha_3. \tag{4.3}$$

Thus we are in position to study the second-order correlation function $g_j^{(2)}(0)$ for the jth mode, measuring the deviation from the Poisson statistics of the state $|\underline{\alpha},\underline{r}\rangle$ [19]. This quantity has been defined by

$$\begin{aligned} g_j^{(2)}(0) &= \frac{\langle \hat{A}_j^{\dagger 2}\hat{A}_j^2\rangle}{\langle \hat{A}_j^{\dagger}\hat{A}_j\rangle^2} \\ &= 1 + \frac{\langle(\Delta\hat{n}_j)^2\rangle - \langle\hat{n}_j\rangle}{\langle\hat{n}_j\rangle^2}, \end{aligned} \tag{4.4}$$

where $\langle(\Delta\hat{n}_j)^2\rangle$ and $\langle\hat{n}_j\rangle$ are the variance and average of the photon number for the jth mode, respectively. It can happen that $g_j^{(2)}(0) = 1$ for Poisson light (*e.g.* coherent states), or $g_j^{(2)}(0) < 1$ for sub-Poisson light (*e.g.* Fock states), otherwise we have super-Poisson light (*e.g.* chaotic field). The degree of coherence $g_j^{(2)}(0)$ can be measured by a set of two detectors.

The mean photon numbers for various modes in three-mode squeezed coherent states are given by

$$\langle\hat{n}_1\rangle_{\rm coh} = |f_1\alpha_1 + f_2\alpha_2^* + f_3\alpha_3^*|^2 + f_2^2 + f_3^2, \tag{4.5a}$$

$$\langle\hat{n}_2\rangle_{\rm coh} = |g_1\alpha_2 + g_2\alpha_3 + g_3\alpha_1^*|^2 + g_3^2, \tag{4.5b}$$

and the photon-number variances are

$$\langle(\Delta\hat{n}_1)^2\rangle_{\rm coh} = (2f_1^2 - 1)\langle\hat{n}_1\rangle_{\rm coh} - (f_1^2-1)^2, \tag{4.6a}$$

$$\langle(\Delta\hat{n}_2)^2\rangle_{\rm coh} = (2g_3^2 + 1)\langle\hat{n}_2\rangle_{\rm coh} - g_3^4, \tag{4.6b}$$

where coh stands for squeezed coherent states. Expressions related with the 3rd mode can be obtained from those of the 2nd mode by using the following transformation

$$(g_1, g_2, g_3) \rightarrow (h_3, h_1, h_2). \tag{4.7}$$

Having obtained equations (4.5, 4.6), we are in position to examine the second-order correlation function given by (4.4). In the following we shall restrict our discussions to the first mode 1, because the other modes would have similar behaviour.

In phase space, squeezed coherent states $|\alpha, r\rangle$ are represented by a noise ellipse with the origin at α, and they do not exhibit sub-Poisson distribution, *i.e.* they exhibit Poisson distribution at $r = 0$ which is growing rapidly to superthermal distribution, *i.e.* $g^{(2)}(0) > 2$, and it persists for a large domain of r [24,25]. In our model of three-mode squeezed coherent states we shall show that they exhibit only partial coherence behaviour, *i.e.* $1 < g^{(2)}(0) < 2$.

The condition for sub-Poissonian statistics is that the variance $\langle(\Delta\hat{n}_j)^2\rangle$ must be less than the mean photon number $\langle\hat{n}_j\rangle$. We show that this condition will not be fulfilled for all modes, *i.e.* sub-Poissonian light cannot be obtained, because we have, *e.g.* for the first mode, the inequality

$$2|f_1^2\alpha_1 + f_2\alpha_2^* + f_3\alpha_3^*|^2 + f_2^2 + f_3^2 < 0, \tag{4.8}$$

which will not be satisfied for any values of the coherent amplitudes α_j.

For the first mode $\hat{A}_1$ we have

$$g_1^{(2)}(r_i) = 1 + \left[\frac{2(f_1^2-1)\langle\hat{n}_1\rangle_{\rm coh} - (f_1^2-1)^2}{\langle\hat{n}_1\rangle_{\rm coh}^2}\right]. \tag{4.9}$$

It is clear when $r_j = 0$, we recover the normalized second-order correlation function for coherent states. To reach Poissonian statistics or thermal statistics, the expectation value of the photon number $\langle\hat{n}_1\rangle$ should have the following values:

$$\langle\hat{n}_1\rangle_{\rm p} = \frac{1}{2}(f_1^2 - 1), \tag{4.10a}$$

$$\langle\hat{n}_1\rangle_{\rm th} = f_1^2 - 1, \tag{4.10b}$$

where subscripts "p" and "th" denote the corresponding quantities for Poisson and thermal distributions, respectively. For instance, to obtain thermal field, from (4.5a) and (4.10b) we have

$$|f_1\alpha_1 + f_2\alpha_2^* + f_3\alpha_3^*|^2 = 0. \tag{4.11}$$

It is clear that (4.11) is satisfied only when $\alpha_j = 0$, *i.e.* for three-mode squeezed vacuum states, and hence super-thermal statistics cannot be available. Similar procedures show that Poissonian statistics cannot be obtained. This is a consequence of intermodal correlations of three-mode squeezed coherent states, for $\alpha_j \neq 0$, light exhibits only partially coherence. This can be seen in Figure 2, where we display the normalized normal second-order correlation function for the first mode against r_1 for $\alpha_j = 2\exp(\mathrm{i}\pi/4), j = 1,2,3$, $r_3 = 0.2$ with $r_2 = 0.4, 0.8$, and 1.5 corresponding to solid, short-dashed, and long-dashed curves, respectively. Also we have displayed the corresponding normalized second-order correlation function for two-mode squeezed coherent state (star-centered curve) for the amplitudes $\alpha_j = 2\exp(\mathrm{i}\pi/4), j = 1,2$, for sake of comparison. In this figure it is clear that partial coherence is dominant, and it persists for large values of r_1 and the amount of fluctuations is sensitive to r_2. However, the normalized second-order correlation function for two-mode squeezed coherent state (star-centered curve) is a growing function, starting from 1 (Poisson distribution) when there is no squeezing $r = 0$, and becomes stable for large values of r displaying partial coherence behaviours. These values exceed those for three-mode squeezed coherent state for large domain of r.

Number states are the energy eigenstates of the free-field Hamiltonian. These states are purely nonclassical states since they are always representing sub-Poissonian light. Nevertheless, they are not squeezed states as well as they carry no information on the phase owing to the number of photons which is definite. We shall discuss the sub-Poisson properties of three-mode squeezed number

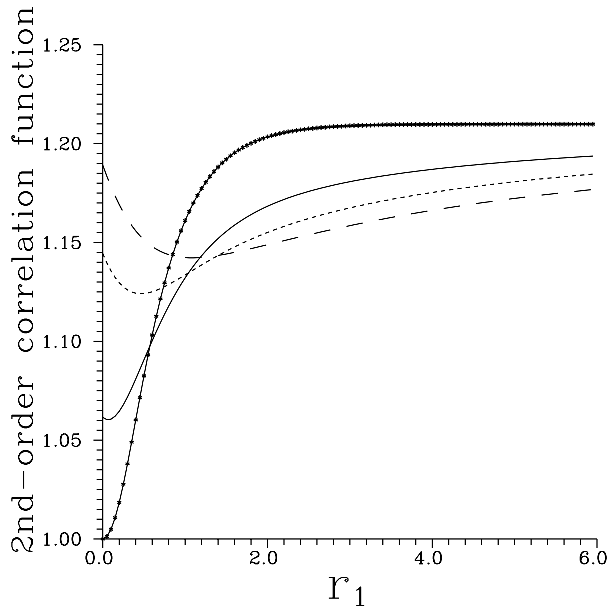


Fig. 2. Normalized normal second-order correlation function $g_1^{(2)}(0)$ (first mode) for three-mode squeezed coherent state against r_1, when $\alpha_j = 2\exp(\mathrm{i}\pi/4)$, $j = 1, 2, 3$, $r_3 = 0.2$, taking $r_2 = 0.4$ (solid curve), 0.8 (short-dashed curve) and 1.5 (long-dashed curve); star-centered curve is corresponding to the second-order correlation function for two-mode squeezed coherent state and $\alpha_j = 2\exp(\mathrm{i}\pi/4)$, $j = 1, 2$.

states. This state can be written with the aid of three-mode squeeze operator as

$$|\underline{n}, \underline{r}\rangle \equiv \hat{S}(\underline{r})|n_1\rangle|n_2\rangle|n_3\rangle, \tag{4.12}$$

for simplicity we shall set $\underline{n} = (n_1, n_2, n_3)$.

The mean photon numbers in three-mode squeezed number state are

$$\langle\hat{n}_1\rangle_n = \bar{n}_1 f_1^2 + (\bar{n}_2 + 1)f_2^2 + (\bar{n}_3 + 1)f_3^2, \tag{4.13a}$$
$$\langle\hat{n}_2\rangle_n = \bar{n}_2 g_1^2 + \bar{n}_3 g_2^2 + (\bar{n}_1 + 1)g_3^2, \tag{4.13b}$$

and the variances of the photon number are

$$\begin{aligned}\langle(\Delta\hat{n}_1)^2\rangle_n = {} & f_1^2 f_2^2[\bar{n}_1 + \bar{n}_2 + 2\bar{n}_1\bar{n}_2 + 1] \\ & + f_1^2 f_3^2[\bar{n}_1 + \bar{n}_3 + 2\bar{n}_1\bar{n}_3 + 1] \\ & + f_2^2 f_3^2[\bar{n}_2 + \bar{n}_3 + 2\bar{n}_2\bar{n}_3],\end{aligned} \tag{4.14a}$$
$$\begin{aligned}\langle(\Delta\hat{n}_2)^2\rangle_n = {} & g_1^2 g_2^2[\bar{n}_3 + \bar{n}_2 + 2\bar{n}_2\bar{n}_3] \\ & + g_1^2 g_3^2[\bar{n}_1 + \bar{n}_2 + 2\bar{n}_1\bar{n}_2] \\ & + g_2^2 g_3^2[\bar{n}_1 + \bar{n}_3 + 2\bar{n}_1\bar{n}_3 + 1],\end{aligned} \tag{4.14b}$$

where $\bar{n}_j$ is the mean photon number for the jth mode. The expression for the third mode can be obtained using (4.7). It is worthwhile to refer to [24–28], where further discussions related to squeezed number states are given. For instance, when the squeezing is not significant, *i.e.* r is small, the normalized second-order correlation function can be less than unity, which indicates that the light field has sub-Poissonian statistics [25]. Furthermore, squeezed vacuum state exhibits super-Poisson statistics (precisely superthermal statistics) for $r \neq 0$ [29]. Here, in contrast to the latter results, we prove that three-mode squeezed number state can exhibit thermal statistics when $\bar{n}_j = 0, j = 1, 2, 3$, *i.e.* for three-mode squeezed vacuum states, otherwise sub-Poissonian statistics or partially coherence behaviour is dominant. We focus our attention to the first mode. To obtain thermal statistics, we need to have

$$\langle(\Delta\hat{n}_1)^2\rangle_n - \langle\hat{n}_1\rangle_n - \langle\hat{n}_1\rangle_n^2 = 0; \tag{4.15a}$$

after minor algebra one finds

$$\bar{n}_1(\bar{n}_1 + 1)f_1^4 + \bar{n}_2(\bar{n}_2 + 1)f_2^4 + \bar{n}_3(\bar{n}_3 + 1)f_3^4 = 0. \tag{4.15b}$$

It is clear that the equality sign is only satisfied for $\bar{n}_j = 0, j = 1, 2, 3$. In other words, superthermal statistics will never occur. This is in contrast with the squeezed number state. In Figure 3a we depict normalized second-order correlation function for single mode case for three-mode squeezed number state against r_2 for $\bar{n}_j = 1, j = 1, 2, 3$ and $(r_1, r_3) = (0.5, 0.3)$. We notice in general that partially coherence behaviour is dominant again. Further we can see sub-Poissonian behaviour for small values of μ for modes 1 and 3 with maximum value at the third one. For sake of comparison, we displayed the normalized second-order correlation functions for single mode (dashed curve) and two-mode (solid curve) squeezed number states in Figure 3b for $\bar{n}_j = 1, j = 1, 2$, where we can see that both of them have sub-Poissonian statistics for lower values of squeeze parameter, otherwise they characterize partially coherence behaviour of light beams.

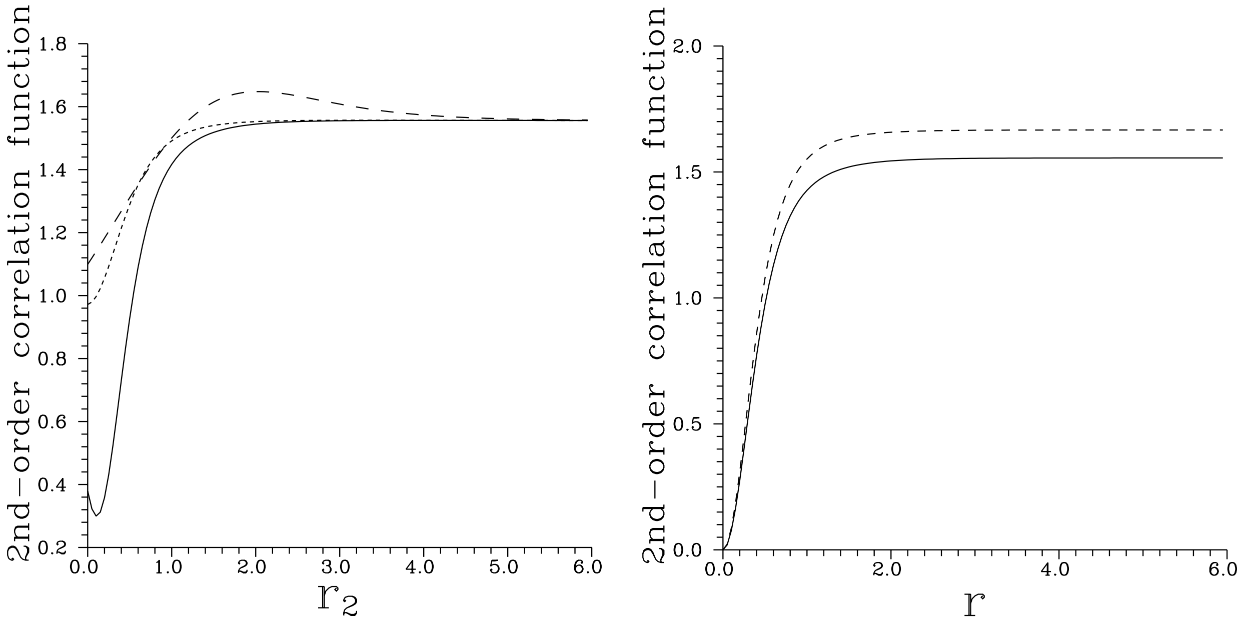
Finally we turn our attention to discuss the effect of intermodal correlations in terms of anticorrelations (antibunching) in three-mode squeezed coherent states. This will be done by two means. The first mean is given by introducing the photon-number operator $\hat{n}_{j,k} = \hat{A}_j^\dagger\hat{A}_j + \hat{A}_k^\dagger\hat{A}_k$ and calculating the quantity

$$\langle(\Delta W_{j,k})^2\rangle = \langle:(\hat{n}_{j,k})^2:\rangle - \langle\hat{n}_{j,k}\rangle^2,$$

where : : denotes the normally ordered operator, *i.e.* creation operators $\hat{A}_j^\dagger$ are to the left of annihilation operators $\hat{A}_j$. The quantum anticorrelation effect is then characterized in terms of the variance of the photon number, which is less than the average of the photon number for nonclassical light, by negative values of $\langle(\Delta W_{j,k})^2\rangle$, *i.e.* negative cross-correlations taken two times are stronger than the sum of quantum noise in single modes [30]. The second way is based on violation of Cauchy-Schwarz inequality. The violation of Cauchy-Schwarz inequality can be observed in a two-photon interference experiment [31]. Classically, Cauchy-Schwarz inequality has the form [32]

$$\langle I_1 I_2\rangle \leq \langle I_1^2\rangle\langle I_2^2\rangle, \tag{4.16a}$$

where $I_j, j = 1, 2$ are classical intensities of light measured by different detectors in a double-beam experiment.



(a) (b)

Fig. 3. Normalized normal second-order correlation function for: (a) three-mode squeezed number states against r_2 for different modes, when $\bar{n}_j = 1, j = 1, 2, 3$, $(r_1, r_3) = (0.5, 0.3)$ for mode 1 (solid curve), mode 2 (short-dashed curve) and mode 3 (long-dashed curve); (b) for single mode squeezed number states (solid curve, $\bar{n}_1 = 1$) and two-mode squeezed number state (dashed curve, $\bar{n}_j = 1, j = 1, 2$) against squeezing parameter r.

In quantum theory, the deviation from this classical inequality can be represented by the factor [33]

$$I_{j,k} = \frac{[\langle \hat{A}_j^{\dagger 2} \hat{A}_j^2 \rangle \langle \hat{A}_k^{\dagger 2} \hat{A}_k^2 \rangle]^{\frac{1}{2}}}{\langle \hat{A}_j^\dagger \hat{A}_j \hat{A}_k^\dagger \hat{A}_k \rangle} - 1. \tag{4.16b}$$

The negative values for the quantity $I_{j,k}$ mean that the intermodal correlation is larger than the correlation between the photons in the same mode [18] and this indicates strong violation of the Cauchy-Schwarz inequality.

To make use of these tools for the three-mode squeezed coherent state, we have to derive the expectation values of cross photon-number operators between various modes, the explicit forms for these quantities are given in Appendix A.

The quantity $\langle (\Delta W_{j,k})^2 \rangle$ can be observed if both the modes are simultaneously detected. In general, photon antibunching can occur in dependence on the values of the parameters r_j and α_j. In Figure 4 we have plotted the quantity $I_{j,k}$ indicating the violation of Cauchy-Schwarz inequality between the jth mode and the kth mode against squeeze parameter r_1, where $(r_2, r_3) = (0.4, 0.2)$ and $\alpha_j = 1, j = 1, 2, 3$ (real). Further, solid, short-dashed, and long-dashed curves correspond to the above quantity obtained between $(1, 2), (2, 3)$, and $(1, 3)$ modes, respectively. In this figure we can observe that all curves can take on negative values reflecting the violation of the inequality, *i.e.* the photons are more strongly correlated than it is possible classically. The strongest violation of this inequality occurs in the $(1, 3)$ mode for lower r_1 and then the curve monotonically increases to positive values for larger r_1. The weakest violation is in the $(2, 3)$ mode for which the curve decreases from positive values to the negative values as r_1 increases. Clearly, the violation of

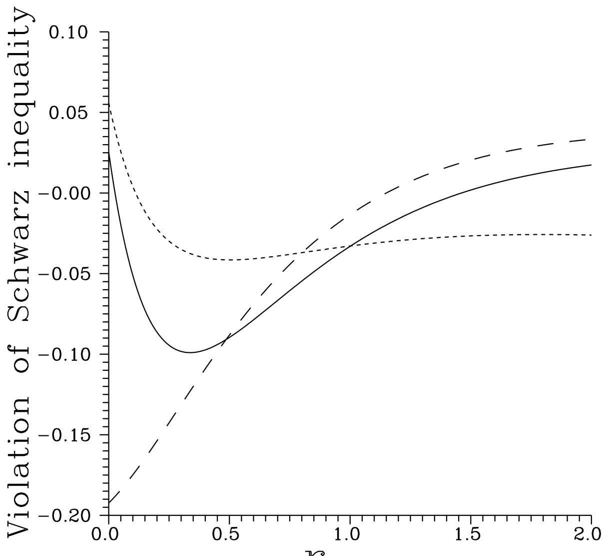


Fig. 4. The quantity $I_{j,k}$, indicating the violation of the Cauchy-Schwarz inequality between the jth mode and kth mode, against squeeze parameter r_1, where $(r_2, r_3) = (0.4, 0.2)$ and $\alpha_j = 1$, $j = 1, 2, 3$ (real). Solid, short-dashed, and long-dashed curve correspond to the above quantity considered for modes $(1, 2)$, $(2, 3)$, and $(1, 3)$, respectively.

this inequality is sensitive to the values of squeeze parameters and coherent amplitudes. As is known, the correspondence between quantum and classical theories can be established *via* Glauber-Sudarshan P-representation. But P-representation does not have all the properties of a classical distribution function, especially for quantum fields. So the violation of the Cauchy-Schwarz inequality provides explicit evidence of the quantum nature of intermodal correlation between modes which imply that the P-distribution function possesses strong quantum properties [33].

5 Quasiprobability distribution functions

In this section we shall extend our work to include the quasiprobability distribution functions: W-Wigner, Q-Husimi and P-Glauber functions. In fact, these functions are important tools to discuss the statistical description of a microscopic system. In other words, the most important considerable quantities for the state of a quantum system are various expectation values of the system operators. These expectation values can be obtained by appropriate integration in phase space with these functions. Further, we find the contours of the W-function giving the variances in the field quadratures, whereas the contours of the Q-function provide the anti-normally ordered variance in the field quadratures. Moreover, these quasidistributions can be determined in homodyne tomography [34].

In the following we shall consider such functions for three different states: three-mode squeezed coherent, number and thermal states. As we have mentioned before, for the correlated system, if the measurement of an observable of subsidiary systems is performed, this projects the other systems into new states; and hence we shall study the quasiprobability distribution functions for single mode case that to demonstrate such effect. Furthermore, our discussions will extend to include phase distributions for single mode states with the aid of Q-function as well as the derivation of joint photon-number distribution for three-mode squeezed coherent state.

In order to evaluate these functions we need to calculate the characteristic function for these desired states. The s-parameterized joint characteristic function for the three-mode system is defined by

$$C^{(3)}(\zeta_1,\zeta_2,\zeta_3,s)=\mathrm{Tr}\left\{\hat{\rho}\exp[\sum_{j=1}^{3}(\zeta_j\hat{A}_j^\dagger-\zeta_j^*\hat{A}_j+\frac{s}{2}|\zeta_j|^2)]\right\}, \quad (5.1)$$

where $\hat{\rho}$ is the density matrix operator for the state under consideration, and s takes on values $1, 0$ and -1 corresponding to normally, symmetrically and antinormally ordered characteristic functions, respectively.

Thus s-parameterized joint quasiprobability functions are given by

$$W^{(3)}(\beta_1,\beta_2,\beta_3,s)=\frac{1}{\pi^6}\iiint C^{(3)}(\zeta_1,\zeta_2,\zeta_3,s)\times\prod_{j=1}^{3}\exp(\beta_j\zeta_j^*-\beta_j^*\zeta_j)\mathrm{d}^2\zeta_j. \quad (5.2)$$

When $s=1,0,-1$, equation (5.2) gives formally Glauber P-function, Wigner W-function and Husimi Q-function, respectively.

On the other hand, the s-parameterized quasiprobability functions for the single mode can be attained by means of integrating two times in the corresponding joint quasiprobability functions or by using the single mode s-parameterized characteristic function. In fact, the characteristic function for mode $\hat{A}_1$ (say) can be obtained from $C^{(3)}(\zeta_1,\zeta_2,\zeta_3,s)$ by simply setting $\zeta_2=\zeta_3=0$, so in the following we shall consider only the s-parameterized joint characteristic function. Hence the s-parameterized single mode quasiprobability can be derived by

$$W^{(1)}(\beta_1,s)=\iint W^{(3)}(\beta_1,\beta_2,\beta_3,s)\mathrm{d}^2\beta_2\mathrm{d}^2\beta_3, \quad (5.3)$$

etc., or by

$$W^{(1)}(\beta_j,s)=\frac{1}{\pi^2}\int C^{(1)}(\zeta_j,s)\exp(\beta_j\zeta_j^*-\beta_j^*\zeta_j)\mathrm{d}^2\zeta_j. \quad (5.4)$$

$C^{(1)}(\zeta_j,s)$ in (5.4) is the s-parameterized single mode characteristic function. The superscripts (1) and (3) in the above equations stand for single mode case and three-mode case, respectively.

5.1 Three-mode squeezed coherent states

Here we study quasiprobability distribution functions for three-mode squeezed coherent state, specified by the density matrix

$$\hat{\rho}=\hat{S}(\underline{r})|\alpha_1,\alpha_2,\alpha_3\rangle\langle\alpha_3,\alpha_2,\alpha_1|\hat{S}^\dagger(\underline{r}). \quad (5.5)$$

Thus substituting (5.5) into (5.1) and after some minor calculations provides the s-parameterized joint characteristic function

$$C^{(3)}_{\mathrm{coh}}(\zeta_1,\zeta_2,\zeta_3,s)=\exp\left[\frac{1}{2}\sum_{j=1}^{3}(s|\zeta_j|^2-|\eta_j|^2)\right]\times\exp[(\alpha_1^*\eta_1-\alpha_1\eta_1^*)+(\alpha_2^*\eta_2-\alpha_2\eta_2^*)+(\alpha_3^*\eta_3-\alpha_3\eta_3^*)], \quad (5.6)$$

where we have used the following abbreviations

$$\eta_1=\zeta_1 f_1-\zeta_2^* g_3-\zeta_3^* h_2, \quad (5.7a)$$
$$\eta_2=\zeta_2 g_1-\zeta_1^* f_2+\zeta_3 h_3, \quad (5.7b)$$
$$\eta_3=\zeta_3 h_1+\zeta_2 g_2-\zeta_1^* f_3. \quad (5.7c)$$

$$W^{(3)}(\beta_1,\beta_2,\beta_3,s) = \frac{2}{\pi^3}\left[\frac{(2f_2^2+2f_3^2+1-s)^{-1}}{\nu_1\nu_2}\right]\exp\left[-\frac{2|\beta_1|^2}{(2f_2^2+2f_3^2+1-s)}\right] \times \exp\left\{\frac{1}{4\nu_1}\left[(A^2-C^2)\cos^2\phi+(B^2-D^2)\sin^2\phi+(CD-AB)\sin(2\phi)\right]\right\} \times \exp\left\{\frac{1}{4\nu_2}\left[(B^2-D^2)\cos^2\phi+(A^2-C^2)\sin^2\phi-(CD-AB)\sin(2\phi)\right]\right\}, \quad (5.8)$$

Having obtained the characteristic function, we are therefore in a position to find the s-parameterized quasiprobability functions for three-mode squeezed coherent state by inserting equation (5.6) into equation (5.2) and evaluating the integral, thus we get

see equation (5.8) above

where

$$\nu_1 = \mu_1\cos^2\phi+\mu_2\sin^2\phi+\mu_3\sin(2\phi), \quad (5.9a)$$

$$\nu_2 = \mu_2\cos^2\phi+\mu_1\sin^2\phi-\mu_3\sin(2\phi), \quad (5.9b)$$

and

$$\phi = \frac{1}{2}\tan^{-1}\left(\frac{2\mu_3}{\mu_1-\mu_2}\right), \quad (5.9c)$$

whereas

$$\mu_1 = \frac{1}{2}(2f_2^2+2f_3^2+1-s)^{-1} \times\left[2f_1^2(1-s)-2g_3^2(1+s)-(1-s^2)\right], \quad (5.10a)$$

$$\mu_2 = \frac{1}{2}(2f_2^2+2f_3^2+1-s)^{-1} \times\left[2f_1^2(1-s)-2h_2^2(1+s)-(1-s^2)\right], \quad (5.10b)$$

and

$$\mu_3 = (2f_2^2+2f_3^2+1-s)^{-1} \times\left[2f_1^2g_2h_1+g_1h_3(1+s)\right]. \quad (5.10c)$$

In equation (5.8) we have also used the following definitions

$$A = (\tilde{\alpha_2^*}-\tilde{\alpha_2})+2f_1g_3(2f_2^2+2f_3^2+1-s)^{-1}(\tilde{\alpha_1^*}-\tilde{\alpha_1}), \quad (5.11a)$$

$$B = (\tilde{\alpha_3^*}-\tilde{\alpha_3})+2f_1h_2(2f_2^2+2f_3^2+1-s)^{-1}(\tilde{\alpha_1^*}-\tilde{\alpha_1}), \quad (5.11b)$$

$$C = (\tilde{\alpha_2^*}+\tilde{\alpha_2})-2f_1g_3(2f_2^2+2f_3^2+1-s)^{-1}(\tilde{\alpha_1^*}+\tilde{\alpha_1}), \quad (5.11c)$$

$$D = (\tilde{\alpha_3^*}+\tilde{\alpha_3})-2f_1h_2(2f_2^2+2f_3^2+1-s)^{-1}(\tilde{\alpha_1^*}+\tilde{\alpha_1}), \quad (5.11d)$$

where $\tilde{\alpha_j} = (\bar{\alpha}_j-\beta_j)$, $j = 1,2,3$ and $\bar{\alpha}_j$ represents the expectation value of the operators $\bar{A}_j$ given by equation (3.1) in the coherent state. The three-mode functions $W^{(3)}(\beta_1,\beta_2,\beta_3,s)$ given in (5.8) are 6-dimensional Gaussian functions and display the nonclassical correlation nature by involving the terms A^2, B^2, etc. when the cross-terms $\beta_1\beta_2, \beta_1\beta_3$, etc. are not zero. Furthermore, we can see that the three-mode P representation does not exist at least for some values of squeezing parameters r_j, *i.e.* if $\nu_1 = 0$ or $\nu_2 = 0$ or both (see Eq. (5.8)), the physical reason for non-conditional breaking down of the P function lies in the extremely strong correlation between the amplitudes of the three modes of the system during the evolution determined by squeeze operator [19]. Such tightly correlation causes that, the modes may no longer fluctuate independently even in small amount that is allowed in a pure coherent states [15].

Now we turn our attention to single mode case, using similar procedure as before. The s-parameterized quasiprobability function takes the form

$$W^{(1)}(\beta_j,s) = \frac{2}{\pi(\tau_j-s)}\exp\left(-2\frac{|\beta_j-\bar{\alpha}_j|^2}{\tau_j-s}\right), \quad (5.12)$$

where $j = 1,2,3$ and

$$\tau_1 = 2f_2^2+2f_3^2+1, \quad \tau_2 = 2g_3^2+1, \quad \tau_3 = 2h_2^2+1. \quad (5.13)$$

Light fields for which the P representation is not a well-behaved distribution will exhibit nonclassical features. Clearly single mode P-function is well defined, because the parametric systems evolution broadens P distribution (increases the radius of the Wigner contour compared to the initial one) [15] and reflects that there is no (single mode) nonclassical behaviour, *e.g.* sub-Poissonian statistics and squeezing. Furthermore, it has been shown that P-function for the superposition of two fields is the convolution of the P-function for each field considered individually [35], so that (5.12) (with $s = 1$) describes the superposition of P-function of a coherent state with complex amplitude $\bar{\alpha}_j$ and P-function of a chaotic mixture with variance $(\tau_j-1)/2$ [19], *i.e.* displaced thermal light. Further, equation (5.12) has a Gaussian form in phase space with width and center are dependent on r_j, α_j with a circular symmetric contour as a result of the two quadrature variances are equal. Consequently the single mode photon-number distribution does not exhibit oscillations, in contrast with this of squeezed coherent state [36], owing to the noise ellipse of W-function which is accompanied with this behaviour is absent.

Squeezed states have phase-sensitive noise properties, and it is particularly interesting to study their phase

$$P(m_1,m_2,m_3)=m_1!m_2!m_3!f_1^{-2}\left[\frac{|\bar{\alpha}_1|}{f_1}\right]^{2m_1}\left[\frac{f_2}{|\bar{\alpha}_1|}\right]^{2m_2}\exp\left(-\sum_{j=1}^{3}|\bar{\alpha}_j|^2\right)$$
$$\times\left|\left|\frac{\bar{\alpha}_3g_1-\bar{\alpha}_2h_3}{f_1}\right|^{2m_3}\exp\left\{2\mathrm{Re}\left[\frac{\bar{\alpha}_1(g_3\bar{\alpha}_2+h_2\bar{\alpha}_3)}{f_1}\right]\right\}\sum_{l=0}^{m_3}[(m_1-l)!(m_3-l)!l!]^{-1}\left[\frac{f_1f_3}{\bar{\alpha}_1(\bar{\alpha}_2h_3-\bar{\alpha}_3g_1)}\right]^{l}L_{m_2}^{m_1-m_2-l}\left[\frac{\bar{\alpha}_1(\bar{\alpha}_2h_1-\bar{\alpha}_3g_2)}{f_1f_2}\right]\right|^2$$
$$m_1\geq l,m_3\geq l \quad (5.18)$$

properties. This can be done by integrating single mode Q-function over the radial variable [37]

$$P(\theta_j)=\int_0^\infty Q(\beta_j)|\beta_j|\mathrm{d}|\beta_j|, \quad (5.14)$$

where $\beta_j=|\beta_j|\exp(\mathrm{i}\theta_j)$ and $Q(\beta_j)$ is given by (5.12) for $s=-1$. Hence, we have for the jth mode

$$P(\theta_j)=\frac{1}{2\pi\sqrt{\tau_j+1}}\exp\left[\frac{b_j^2-4|\bar{\alpha}_j|^2}{2(\tau_j+1)}\right]\times\left\{\sqrt{(\tau_j+1)}\exp\left[-\frac{b_j^2}{2(\tau_j+1)}\right]+\frac{b_j\sqrt{\pi}}{2}\left[1+\mathrm{erf}\left(\frac{\sqrt{2}b_j}{\sqrt{\tau_j+1}}\right)\right]\right\}, \quad (5.15a)$$

where

$$b_j=\bar{\alpha}_j\exp(-\mathrm{i}\theta_j)+\bar{\alpha}_j^*\exp(\mathrm{i}\theta_j), \quad (5.15b)$$

with $j=1,2,3$, $\bar{\alpha}_j$ have the same meaning as before, and we have used the Gauss error function, which is defined by the well-known formula

$$\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x\exp(-y^2)\mathrm{d}y. \quad (5.16)$$

For the squeezed coherent state, authors of [38,39] found that for real $\alpha>\mathrm{e}^{2r}$ the phase distribution contained only one peak, but for a small displacement, a bifurcation in the phase space distribution was obtained. This bifurcation phenomenon is connected with the two-photon nature of squeeze operator (1.1), and it has been investigated in [37] showing that a competition arises between the two-peak structure of the squeezed vacuum and the single peak structure of the coherent state. In our model, restricted ourselves to real α_j, *i.e.* $b_j=2\bar{\alpha}_j\cos\theta$, detailed examination to the formula (5.15a) shows that it is a 2π-periodic function, a symmetric function ($P(-\theta)=P(\theta)$) around $\theta=0$ and also it has its maximum height at $\theta=0$. Moreover, this formula has a similar structure as that for coherent states (which can be recovered from (5.15a) by setting $r_j=0$). This means that the phase distribution of the single mode exhibits a one-peak structure for all values of α_j and r_j (see Fig. 5 for shown values of parameters). That is the phase distribution of the single mode case, as output from three-mode squeezed coherent state, is insensitive to the quantum correlations between the systems. This situation is the same as that of two-mode squeezed coherent state, *i.e.* when $r_2=r_3=0$ [40]. However, it has been shown that the joint phase distribution for the two-mode squeezed vacuum depends only on the sum of the phases of the two modes, and that the sum of the two phases is locked to a certain value as the squeezing parameter increases [40].

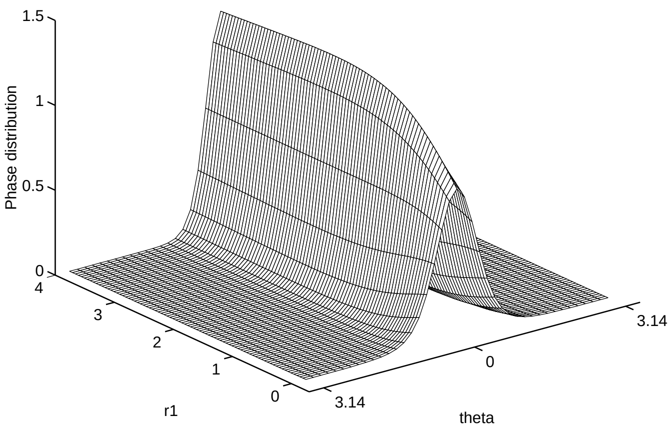


Fig. 5. Phase distribution $P(\theta,r_1)$ against θ and r_1 for the first mode as output from three-mode squeezed coherent state for $\alpha_j=1$, $j=1,2,3$, $r_2=1$ and $r_3=0.5$.

Now we conclude this subsection by deriving the joint photon-number distribution for three-mode squeezed coherent state using the joint P-function [19] as

$$P(m_1,m_2,m_3)=\langle m_1,m_2,m_3|\hat{\rho}|m_3,m_2,m_1\rangle=\iiint W(\beta_1,\beta_2,\beta_3,s=1)\times\prod_{j=1}^{3}\frac{|\beta_j|^{2m_j}}{m_j!}\exp(-|\beta_j|^2)\mathrm{d}^2\beta_j, \quad (5.17)$$

by substituting (5.8) into (5.17), performing the integrations then we have

see equation (5.18) above

where $\bar{\alpha}_j$ is the expectation value of the operator $\bar{A}_j$ with respect to the coherent state. It is clear that from the form (5.18), when $r_j=0$, the three single photon-number distributions for the initial coherent states appear multiplied by each other. In Figure 6 we have plotted the diagonal joint photon-number distribution, *i.e.* taking

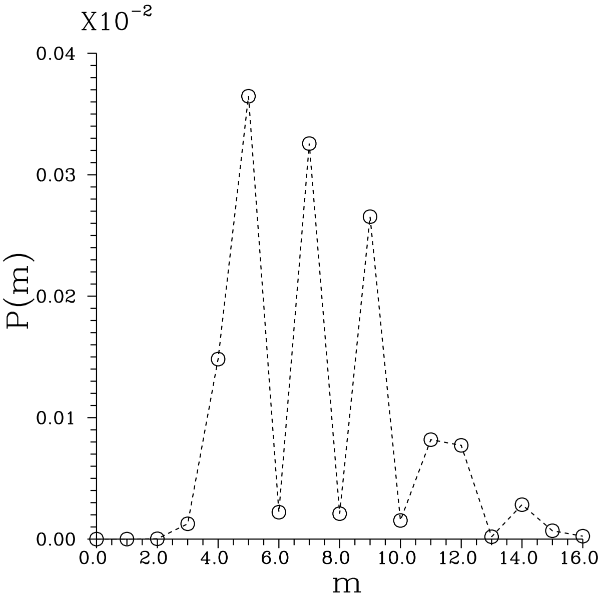


Fig. 6. The diagonal joint photon-number distribution $P(m)$ against m for three-mode squeezed coherent state for $(\alpha_1, \alpha_2, \alpha_3) = (2, 2, 3)$ and $(r_1, r_2, r_3) = (1.8, 0.2, 0.1)$. Circle points represent the values of $P(m)$ corresponding to the values of m.

$m_1 = m_2 = m_3 = m$, for the shown values of the parameters. We may point out that, since the curve shown in this figure is not continuous where m takes on nonnegative values, therefore we have used the dashed curve to join the different values of $P(m)$ (circle points) to visualize the behaviour of the photon-number distribution. From this figure it is clear that this distribution can exhibit nonclassical oscillations as those of the single-mode squeeze operator [36] as well as two-mode squeeze operator [41]. For these two cases the oscillations have been investigated as the interference in phase space. We have shown earlier from the investigation of the W-function that these oscillations are absent from the marginal distributions for each mode. So we can conclude, the strong quantum correlation between different modes is the source of these nonclassical oscillations as well as of the squeezing property.

5.2 Three-mode squeezed number state

Here we calculate the same quantities, as in Section 5.1, for three-mode squeezed number state (4.12). Therefore s-parametrized joint characteristic function and W-function can be written as follows

$$C_{sn}^{(3)}(\zeta_1, \zeta_2, \zeta_3, s) = \prod_{j=1}^{3} \exp(-\frac{1}{2}|\eta_j|^2 + \frac{s}{2}|\zeta_j|^2) L_{n_j}(|\eta_j|^2), \tag{5.19}$$

$$W_{sn}^{(3)}(\beta_1, \beta_2, \beta_3, s=0) = \left(\frac{2}{\pi}\right)^3 \tilde{k}(-1)^{n+m+l} \times \prod_{j=1}^{3} \exp[-2|\gamma_j|^2] L_{n_j}(4|\gamma_j|^2), \tag{5.20}$$

where

$$\gamma_1 = \beta_1 f_1 - \beta_2^* g_3 - \beta_3^* h_2, \tag{5.21a}$$
$$\gamma_2 = \beta_2 g_1 + \beta_3 h_3 - \beta_1^* f_2, \tag{5.21b}$$
$$\gamma_3 = \beta_3 h_1 + \beta_2 g_2 - \beta_1^* f_3, \tag{5.21c}$$

while η_j are defined by (5.7) and

$$\tilde{k} = [f_1(g_1 h_1 - g_2 h_3) + f_2(g_2 h_2 - g_3 h_1) + f_3(g_3 h_3 - g_1 h_2)]^2, \tag{5.22}$$

where $L_k(x)$ are the Laguerre polynomials of order k:

$$L_k(x) = \sum_{j=0}^{k} (-1)^j \frac{k!}{(j!)^2 (k-j)!} x^j. \tag{5.23}$$

Intermodal correlation is evident in the expression (5.20) for W-function, where cross-terms $\beta_1\beta_2$, etc. are present.

Similarly single mode s-parametrized quasiprobability function for mode 1 is

$$W_{sn}^{(1)}(\beta_1, s) = \frac{2}{\pi(\tau_1 - s)} \times \prod_{j=1}^{3} L_{n_j}\left[f_j^2 \frac{\partial^2}{\partial\beta_1 \partial(-\beta_1^*)}\right] \exp\left[-\frac{2|\beta_1|^2}{(\tau_1 - s)}\right], \tag{5.24}$$

where τ_1 is given in (5.13). Expressions for the second mode and third mode can be obtained from (5.24) by replacing f_j functions by g_j and h_j functions, respectively.

Wigner function can be used to trace the nonclassical behaviour of the light fields, which means that it can take on negative values for some states and this is generally regarded as a signature of nonclassical effects. Here we discuss the behaviour of W-function for the first mode when it is in the Fock state $|1\rangle$ while the other modes are in vacuum. In fact, W-function of squeezed Fock state $|1\rangle$ is well-known by inverted hole which is stretched as a consequence of squeezing parameter [25]. In the following we can use $L_1(x) = 1 - x$ to analyze the behaviour of the model under discussion. For this case it is clear that, from equation (5.24), W-function can exhibit negative values only inside the circle $|\beta_1|^2 < (2f_1^2 - 1)/4f_1^2$ with center at the origin. However, the maximum value will be established at the circle $|\beta_1|^2 = (4f_1^4 - 1)/4f_1^2$. Hence, W-function will not exhibit stretching since the variables $\mathrm{Re}\beta_1$ and $\mathrm{Im}\beta_1$ are not involving squeezing factor which plays an essential role for stretched quadratures of squeezed number state. As a result the radii of these circles are dependent on squeeze parameters r_j, so the negative values of W-function will be sensitive to the values of squeeze parameters, *i.e.* the function has negative values for a range of $|\beta_1|$ shorter than for individual Fock state $|1\rangle$, where it exhibits negative values inside the circle $|\beta_1|^2 < 1/2$. This can be recognized if one compares the well-known shape of W-function of Fock state $|1\rangle$ with that of the single mode case as output from three-mode squeezed number states. For this purpose Figure 7 is displayed for W-function of $|1, 0, 0\rangle$ against $x = \mathrm{Re}\beta_1$

$$Q_{\rm th}^{(3)}(\beta_1,\beta_2,\beta_3)=\frac{\tilde{k}}{\pi^3}\left[\delta_1\delta_2(\bar{n}_1+f_1^2)\right]^{-1}\exp\left[-\frac{|\beta_1|^2}{(\bar{n}_1+f_1^2)}\right]\exp\left\{-\frac{1}{\delta_1}[|\lambda|^2\cos^2\bar{\phi}+|\gamma|^2\sin^2\bar{\phi}-\frac{1}{2}(\lambda\gamma^*+\gamma\lambda^*)\sin(2\bar{\phi})]\right\}$$
$$\times\exp\left\{-\frac{1}{\delta_2}[|\gamma|^2\cos^2\bar{\phi}+|\lambda|^2\sin^2\bar{\phi}+\frac{1}{2}(\lambda\gamma^*+\gamma\lambda^*)\sin(2\bar{\phi})]\right\}, \quad (5.29)$$

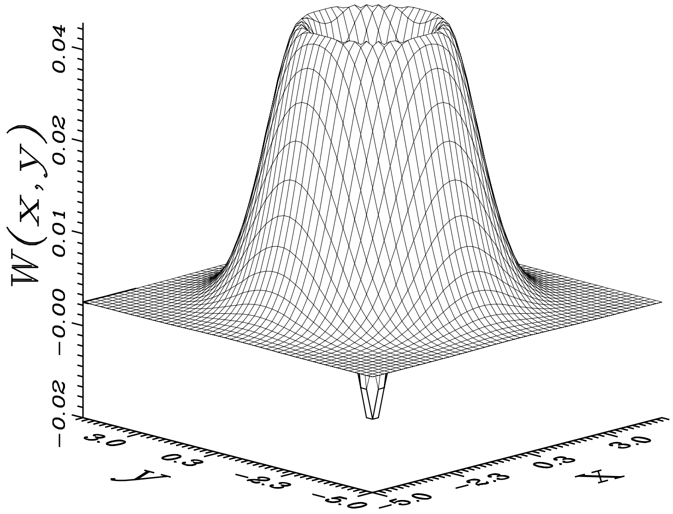


Fig. 7. W-function for the single mode (mode 1) as output from three-mode squeezed number state, assuming the first mode is $|1\rangle$ and the other modes are in vacuum for $(r_1,r_2,r_3)=(0.8,0.9,0.6)$.

and $y=\mathrm{Im}\beta_1$ for $(r_1,r_2,r_3)=(0.8,0.9,0.6)$. We can see that the well-known negative values of W-function of Fock state $|1\rangle$ are smoothed out, in turn, the circle of negative values is enlarged, as we have shown earlier.

The phase distribution for the jth mode, which is in Fock state $|n_j\rangle$ while the other modes are in vacuum states, evolves under the action of three-mode squeezed operator (2.5), with the aid of Q-function, to

$$P(\theta_j)=\frac{1}{2\pi}, \quad (5.25)$$

which is uniform distribution. In other words, in spite of the system is highly correlated the phase distribution in the single mode of purely nonclassical state is phase insensitive. This agree with the fact that the states are represented by a density matrix which is diagonal in the number state basis, have random phase distribution [42].

5.3 Three-mode squeezed thermal states

In a real physical situation no quantum-mechanical system can be totally isolated, *i.e.* any quantum system interacts with its environment (for instance, it interacts with reservoirs), so we cannot avoid noise which can be transferred from the thermal bath to a system and which causes that signal beams are accompanied by thermal fluctuations. Furthermore, there is possibility to generate squeezed thermal light in a microwave Josephson-junction parametric oscillator [43]. So that examination of quantum beam with thermal noise is an important problem from both theoretical and practical points of view. Presently, we consider three-mode squeezed thermal states described by the density matrix

$$\hat{\rho}_T(0)=\frac{1}{(\bar{n}_1+1)(\bar{n}_2+1)(\bar{n}_3+1)}\times\sum_{l,n,m=0}^{\infty}Z_1^lZ_2^nZ_3^m\hat{S}(\underline{r})|n\rangle_1|m\rangle_2|l\rangle_3\times{}_3\langle l|_2\langle m|_1\langle n|\hat{S}^\dagger(\underline{r}), \quad (5.26)$$

where $Z_j=\bar{n}_j/(\bar{n}_j+1), j=1,2,3$ is the quotient of Bose-Einstein (geometric) distribution and $\bar{n}_j, j=1,2,3$ is the jth mode average thermal photon number. As is known thermal distribution has a diagonal expansion in terms of Fock states. This diagonality causes the electric field expectation value vanishes in thermal equilibrium. In fact, this situation is still valid for three-mode squeezed thermal states owing to the linearity of the electric field in the creation and the annihilation operators.

Now s-parametrized joint characteristic function, W-function and Q-function for three-mode squeezed thermal state, respectively, are

$$C_{\rm th}^{(3)}(\zeta_1,\zeta_2,\zeta_3,s)=\exp\left\{-\sum_{j=1}^{3}\left[\left(\bar{n}_j+\frac{1}{2}\right)|\eta_j|^2-\frac{s}{2}|\zeta_j|^2\right]\right\}, \quad (5.27)$$

$$W_{\rm th}^{(3)}(\beta_1,\beta_2,\beta_3)=\frac{\tilde{k}}{\pi^3}\left[\left(\bar{n}_1+\frac{1}{2}\right)\left(\bar{n}_2+\frac{1}{2}\right)\left(\bar{n}_3+\frac{1}{2}\right)\right]^{-1}\times\exp\left[-\sum_{j=1}^{3}\left(\frac{|\beta_j|^2}{\bar{n}_j+\frac{1}{2}}\right)\right], \quad (5.28)$$

see equation (5.29) above

δ_1 and δ_2 in the above equations are given by

$$\delta_1 = \gamma_1 \cos^2 \bar{\phi} + \gamma_2 \sin^2 \bar{\phi} - \gamma_3 \sin(2\bar{\phi}), \quad (5.30a)$$

$$\delta_2 = \gamma_2 \cos^2 \bar{\phi} + \gamma_1 \sin^2 \bar{\phi} + \gamma_3 \sin(2\bar{\phi}), \quad (5.30b)$$

where the quantities λ, γ andγ_j, with $j = 1, 2, 3$ are

$$\lambda = \beta_2 + \beta_1^* \frac{f_1 f_2}{(\bar{n}_1 + f_1^2)}, \quad (5.31a)$$

$$\gamma = \beta_3 + \beta_1^* \frac{f_1 f_3}{(\bar{n}_1 + f_1^2)}, \quad (5.31b)$$

$$\gamma_1 = (\bar{n}_1 + 1) + \frac{\bar{n}_1 f_2^2}{(\bar{n}_1 + f_1^2)}, \quad (5.31c)$$

$$\gamma_2 = (\bar{n}_3 + 1) + \frac{\bar{n}_1 f_3^2}{(\bar{n}_1 + f_1^2)}, \quad (5.31d)$$

and

$$\gamma_3 = \frac{\bar{n}_1 f_2 f_3}{(\bar{n}_1 + f_1^2)}, \quad (5.31e)$$

while

$$\bar{\phi} = \frac{1}{2} \tan^{-1} \left(\frac{2\gamma_3}{\gamma_2 - \gamma_1} \right). \quad (5.31f)$$

Similarly the single mode s-parametrized quasiprobability function is

$$W_{\rm th}^{(1)}(\beta_j, s) = \frac{2}{\pi(\tau_j - s + 2\bar{n}_j)} \exp\left(-\frac{2|\beta_j|^2}{\tau_j - s + 2\bar{n}_j} \right), \quad (5.32)$$

where τ_j are given by (5.13) and $\bar{n}_j$ is the mean photon number for the jth mode in three-mode squeezed thermal states. It is clear that P function is well behaved respecting super-Poissonian statistics in single modes. Comparing equations (5.12, 5.32), we can have both W-functions for single mode coherent and thermal states, as outputs from physical system described by three-mode squeezed operator, which are Gaussian functions, but the center of the former is shifted from the origin and the contour of the latter is broader, since thermal photons have tendency to bunch reflecting Bose-Einstein statistics.

Concerning phase distribution, for the jth mode of thermal squeezed state, we get the same formula as (5.25),where states that are represented by a density matrix which is diagonal in the number state basis, have random phase distribution [42].

6 Conclusions

In this article we have introduced new type of multidimensional squeeze operator which is more general than usually used and which includes two different squeezing mechanisms. This operator arises from the time-dependent evolution operator for the Hamiltonian representing mutual interaction between three different modes of the field. The origin of the nonclassical effects of this operator model is the correlation between the systems where we have shown that the quadratures squeezing and nonclassical oscillations in the photon number distribution can occur in the combination systems rather than in the individual systems.

The quantum statistical properties corresponding to this operator have been traced by means of the variances of the photon-number sum and difference, squeezing phenomenon, Glauber second-order correlation function, violation of Cauchy-Schwarz inequality, quasiprobability distribution functions, joint photon-number distribution and phase probability distribution, considered for three-mode coherent and number states.

For three-mode squeezed coherent state, we found that the second-order correlation function describes partially coherent field, so that one mechanism of squeezing is always surviving, which can be demonstrated also by means of quasiprobability distribution function in single modes. Nevertheless, this behaviour is in contrast with behaviour of squeezed coherent states, where second-order correlation function can display superthermal statistics, *i.e.* $g^{(2)}(0) > 2$. We have found strong violation for Cauchy-Schwarz inequality in some modes, *i.e.* the photons are more strongly correlated than it is allowed classically. Concerning the single-mode phase distribution, single peak structure is dominant for all values of parameters provided that coherent amplitudes are real.

For three-mode squeezed number states, the second-order correlation function is in agreement with that for the squeezed number state and in general it exhibits partial coherence; however, sub-Poissonian behaviour is attained for small values of μ and the maximum value is obtained only for $r_j = 0, j = 1, 2, 3$. Nevertheless, it cannot exhibits superthermal statistics. The signature of the correlations between the three modes appears straightforwardly in the form of quasiprobability functions. Also the range of negative values of the W-function, for a single mode, are highly sensitive to squeeze parameters. Phase distribution, for a single mode, is insensitive to correlation between modes, and it displays a uniform form.

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Appendix A

In this appendix we give the explicit forms for the expectation values of cross photon-number operators between

$$
\begin{aligned}
\langle \hat{n}_1 \hat{n}_2 \rangle = {} & f_1^2 g_3^2 (|\alpha_1|^4 + 2|\alpha_1|^2) \\
& + |(\alpha_2 f_2 + \alpha_3 f_3)(\alpha_2 g_1 + \alpha_3 g_2)|^2 \\
& + (f_1^2 |\alpha_1|^2 + f_1^2 - 1)|\alpha_2 g_1 + \alpha_3 g_2)|^2 \\
& + (|\alpha_1|^2 + 1)\Big[(|\alpha_2|^2 + 1) f_2 g_3 (g_1 f_1 + g_3 f_2) + +(|\alpha_3|^2 + 1) f_3 g_3 (g_2 f_1 + g_3 f_3) \\
& + f_1 g_3 (f_2 g_1 |\alpha_2|^2 + f_3 g_2 |\alpha_3|^2) + (\alpha_1 \alpha_2 + \alpha_1^* \alpha_2^*) f_1 g_3 (f_1 g_1 + f_2 g_3) \\
& + (\alpha_2 \alpha_3^* + \alpha_2^* \alpha_3) g_3 (f_1 f_3 g_1 + f_1 f_2 g_2 + f_2 f_3 g_3) + (\alpha_1 \alpha_3 + \alpha_1^* \alpha_3^*) f_1 g_3 (f_1 g_2 + f_3 g_3)\Big] \\
& + f_1 |\alpha_2 g_1 + \alpha_3 g_2|^2 [(\alpha_1 \alpha_2 + \alpha_1^* \alpha_2^*) f_2 + (\alpha_1 \alpha_3 + \alpha_1^* \alpha_3^*) f_3] \\
& + g_3 |\alpha_2 f_2 + \alpha_3 f_3|^2 [(\alpha_1 \alpha_2 + \alpha_1^* \alpha_2^*) g_1 + (\alpha_1 \alpha_3 + \alpha_1^* \alpha_3^*) g_2] \\
& + (\alpha_1 \alpha_2 + \alpha_1^* \alpha_2^*) h_2 [f_1 f_2 h_2 + f_1^2 h_3 - h_3] \\
& + (\alpha_1 \alpha_3 + \alpha_1^* \alpha_3^*) h_2 [f_1 f_2 h_2 + f_1^2 h_1 - h_2] \\
& + f_1 h_2 [(\alpha_1^2 + \alpha_1^{*2}) |\alpha_2|^2 f_2 h_3 + (\alpha_1^2 + \alpha_1^{*2}) |\alpha_3|^2 f_3 h_1 \\
& + (\alpha_1^2 \alpha_2 \alpha_3 + \alpha_1^{*2} \alpha_2^* \alpha_3^*)(h_1 f_2 + h_3 f_3)]
\end{aligned} \tag{A.1a}
$$

$$
\begin{aligned}
\langle \hat{n}_2 \hat{n}_3 \rangle = {} & h_2^2 g_3^2 [(|\alpha_1|^2 + 2)^2 - 3] + (|\alpha_1|^2 + 1)\Big[g_3^2 |h_1 \alpha_3 + h_3 \alpha_2|^2 \\
& + h_2^2 |g_1 \alpha_2 + g_2 \alpha_3|^2 + h_2 h_3 g_1 g_3 (2|\alpha_2|^2 + 1) \\
& + h_1 h_2 g_2 g_3 (2|\alpha_3|^2 + 1) + h_2 g_3 (g_1 h_2 + g_3 h_3)(\alpha_1 \alpha_2 + \alpha_1^* \alpha_2^*) \\
& + h_2 g_3 (g_3 h_1 + g_2 h_2)(\alpha_1 \alpha_3 + \alpha_1^* \alpha_3^*) \\
& + h_2 g_3 (g_2 h_3 + g_1 h_1)(\alpha_2 \alpha_3^* + \alpha_2^* \alpha_3)\Big] \\
& + |(g_1 \alpha_2 + g_2 \alpha_3)(h_1 \alpha_3 + h_3 \alpha_2)|^2 \\
& + g_3 |h_1 \alpha_3 + h_3 \alpha_2|^2 [g_1 (\alpha_1 \alpha_2 + \alpha_1^* \alpha_2^*) + g_2 (\alpha_1 \alpha_3 + \alpha_1^* \alpha_3^*)] \\
& + h_2 |g_1 \alpha_2 + g_2 \alpha_3|^2 [h_1 (\alpha_1 \alpha_3 + \alpha_1^* \alpha_3^*) + h_3 (\alpha_1 \alpha_2 + \alpha_1^* \alpha_2^*)] \\
& + h_2 g_3 (h_1 g_3 + h_2 g_2)(\alpha_1 \alpha_3 + \alpha_1^* \alpha_3^*) \\
& + h_2 g_3 (h_3 g_3 + h_2 g_1)(\alpha_1 \alpha_2 + \alpha_1^* \alpha_2^*) \\
& + h_2 g_3 (h_1 g_1 + h_3 g_2)(\alpha_1^2 \alpha_2 \alpha_3 + \alpha_1^{*2} \alpha_2^* \alpha_3^*) \\
& + h_2 h_3 g_1 g_3 (\alpha_1^2 \alpha_2^2 + \alpha_1^{*2} \alpha_2^{*2}) + h_1 h_2 g_1 g_3 (\alpha_1^2 \alpha_3^2 + \alpha_1^{*2} \alpha_3^{*2})
\end{aligned} \tag{A.1b}
$$

---

various modes for three-mode squeezed coherent states as

see equations (A.1a, A1b) above.

Corresponding relation between modes 1 and 3 can be obtained from (A.1a) using the transformation (4.7).

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